Deterministic Walks

Kent Quanrud

November 3, 2020

1 Introduction

Many of our recent discussions have been about random walks in graphs. We have seen that random walks are both useful and, with the help of some mathematical tools, analyzable as well. Today's discussion is titled *deterministic walks*. Compared to random walks, there would seem to be no mystery or potential or interest in deterministic walks. Surely we all know how to deterministically walk around a graph. Thus let us clarify, or rather rephrase, our topic of discussion to *derandomizing random walks*.

More precisely, consider the (s, t)-connectivity problems in undirected graphs. Let $G = (V, E)$ be an undirected graph with m edges and n vertices. Let $s, t \in V$. We want to know if s and t are connected in G. This is very easy in normal settings but instead we asked if one could do it in *logarithmic space*. In particular we cannot mark the vertices as we visit them. Surprisingly we *could* decide reachability in logarithmic space, by randomly walking around from s and answering yes if we reach t within the first $O(mn)$ steps. This was connected to the *cover time*: via an unexpected detour through electrical networks, we showed that the excepted number of steps until a random walk from s visits *every* vertex (connected to s) is $\leq 2mn$. A follow up question in the homework showed that if s and t are connected by a path of k edges, then the excepted number of steps is $O(mk)$.

We have also established, in previous discussions, that random walks have stationary distributions, which arise as the unique eigenvector of eigenvalue 1 of the random walk map $R : \mathbb{R}^V \to \mathbb{R}^V$. For undirected graphs, we showed that the convergence rate is connected to the *spectral gap* γ of the random walk matrix - that is, the difference between the maximum eigenvalue 1 and the absolute value of any other eigenvalue of the random walk matrix. Let $d \in \mathbb{N}^V$ be the degrees of G, and recall that the stationary distribution (for undirected graphs) is proportional to d. Since the sum of degrees is $2m$, the stationary distribution is $d/2m$. Let $x_k \in \Delta^V$ denote the distribution after k random steps (from some arbitrary initial distribution $x_0 \in \Delta^V$). Then x_k converged to s at the rate of

$$\left\| x_k - \frac{d}{2m} \right\| \leq (1 - \gamma)^k n.$$

To help interpret the Euclidean norm, keep in mind that $\left\| x_k - \frac{d}{2m} \right\|_\infty \leq \left\| x_k - \frac{d}{2m} \right\|$.

If the spectral gap γ is a constant - say, for the sake of discussion, $\gamma = 1/2$ - then we converge at an exponentially fast raterate:

$$\left\| x_k - \frac{d}{2m} \right\| \leq \frac{n}{2^k}.$$

For $k = O(\log n)$ steps, we have

$$\left\| x_k - \frac{d}{2m} \right\|_\infty \leq \left\| x_k - \frac{d}{2m} \right\| \leq \frac{n}{2^k}.$$

Since $d(v)/2m \geq 1/2m$ for all v, we have $x_k(v) > 0$ for $k \geq O(\log n)$. Note that this holds for any initial distribution. Let's say we started a single vertex s. Then $x_k(v) > 0$ implies, in particular, that *there exists a path form s to v of length k.* That is, any graph with constant spectral graph has $O(\log n)$ diameter. In the context of our (s,t)-connectivity,

To push this thought experiment further, suppose also that G had maximum degree at most a constant, say, 10. In particular, it only takes a constant number of random bits to make each random step. If there is a path from s to t of length $O(\log n)$ steps, then there are $O(\log n)$ random bits that tell us how to get there. A random walk is like guessing these random bits. But when there's only $O(\log n)$ bits to guess, we can enumerate and try all $2^{O(\log n)} = \text{poly}(n)$ possible bit strings!

Observation 1. *(s,t)-connectivity in a constant degree expander can be decided* deterministically *in $O(\log n)$ space.*

Our goal is to prove the following theorem due to Reingold [Rei08]. It asserts that we can deterministically decide (s,t)-connectivity in logarithmic space on *any* graph, not just expanders.

Theorem 2 ([Rei08])**.** *There is a $O(\log n)$ space, polynomial time deterministic algorithm for (s,t)-connectivity in undirected graphs with n vertices.*

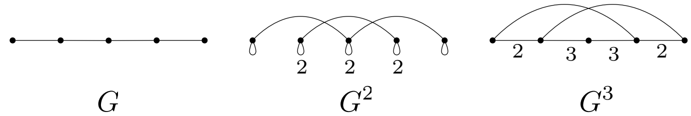
2 An overview of the deterministic connectivity algorithm

We first give a high level of the algorithm and analysis of Theorem 2. We will identify some key technical lemma's, and assume them for the moment and give the rest of the proof Theorem 2. We will prove the lemma's in subsequent sections.

The approach taken by [Rei08] starts from Observation 1 that deterministic log-space connectivity is easy on expander graphs. While of course not all graphs are expanders, [Rei08] applied a sequence of graph transformations (building on previous work such as [RVW02]) that turn the input graph (implicitly) into an expander. We can then deterministically simulate the random walk algorithm on this expander by enumeration.

2.1 Two operations transformations on graphs

2.1.1 Powering



Let $G = (V,E)$ be an undirected graph with m edges and n vertices. For $k \in \mathbb{N}$, the ***k*th power** is the multi-graph on V generated by all the k-step walks for G. That is, for $u, v \in V$, the edge (u,v) has multiplicity equal to the number of walks from

Observe that if R is the random walk matrix on G, then R^k is the random walk matrix on G^k. Recall from before that taking multiple steps along a random walk amplifies the spectral gap. Intuitively, powering improves the spectral gap, but increases the degree.

Lemma 3. *Let $G = (V, E)$ be a regular undirected graph n vertices and degree d. Then G^k is a regular undirected graph on V with degree d^k, with random walk map R^k. If R has spectral gap γ, then R^k has spectral gap $1 - (1-\gamma)^k$.*

The proof of the lemma is implicit in our previous discussion on convergence rates on undirected graphs.

2.1.2 Zig-Zag Product

Our second operation operation is called the **zig-zag product**. The goal of the zig-zag product is to reduce the degree of the graph.

Let $G = (V, E)$ be a regular undirected graph n vertices and degree d. The high level goal is to reduce the degree d. Let $H = (V_0, E_0)$ be a regular undirected graph with d vertices and degree d_0. Morally, d_0 is a universal constant. Note that the number of vertices in H matches the degree of G exactly.

Each vertex in H can be interpreted as a choice of neighbor in G. Speaking very informally, rather than walking on G directly (which requires $\log(d)$ bits per step), we walk on H (which requires only $\log(d_0)$ bits per step), and use the locations in H to induce a walk on G. In this spirit, we identify the vertice of H with the set of indices $[d] = \{1, \ldots, d\}$.

The *zig-zag product*, denoted $\mathcal{Z}(G \,|\, H)$, is a regular graph with vertex set $V_1 \times V_2$ and degree d^2. Each edge in $\mathcal{Z}(G \,|\, H)$ consists of a step in H (with d_0 degrees of freedom), a predetermined "zig-zag" step (with 0 degrees of freedom), followed by a step in H (with d_0 degrees of freedom). For precisely, let $(v_1, i_1) \in V_1 \times V_2$. For $(k_1, k_2) \in [d_0] \times [d_0]$, the (k_1, k_2)th neighbor of (v_1, i_1) is the point (v_2, i_4) obtained by the following steps.

$$\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} \xrightarrow[\text{(a)}]{} \begin{pmatrix} v_1 \\ i_2 \end{pmatrix} \xrightarrow{\text{(b)}} \begin{pmatrix} v_2 \\ i_2 \end{pmatrix} \xrightarrow[\text{(c)}]{} \begin{pmatrix} v_2 \\ i_3 \end{pmatrix} \xrightarrow[\text{(d)}]{} \begin{pmatrix} v_2 \\ i_4 \end{pmatrix}$$

(a) is a step in H from i_1 to its k_1th neighbor, i_2. (b) moves v_1 to the i_2th outgoing neighbor of v_1, v_2. (c) uses v_1 and v_2 to move from i_2 to i_3; namely, we move from i_2 to i_3 if v_1 is the i_3th incoming neighbor of v_2. (d) moves i_3 to its k_2th neighbor in H.

Note that in the move from (v_1, i_1) to (v_2, i_4), v_1 and v_2 are neighbors in G. However, i_1 and i_4 are not (necessarily) neighbors in H! The discontinuity arises in the third step, where the relationship between v_1 and v_2 in G is used to "teleport" i_2 to i_3. Fortunately, all we will really care about is preserving the connectivity in G. If $s, t \in V$ are connected in G, then for each index $i \in [d]$, there is an index $j \in [d]$ such that s_i and t_j are connected. Conversely, if (s, i) and (t, j) are connected in $\mathcal{Z}(G \,|\, H)$, then s and t are connected in G.

Intuitively, the zig-zag product uses a walk on H to implicitly define a constant degree walk on G. In subsequent applications, H will have constant spectral gap, which intuitively means that a random step in H is close to a uniformly random vertex in v.

We defer the proof of the following lemma until Section 3.

Lemma 4. *Let $G = (V, E)$ be a regular undirected graph with n vertices and degree d, with spectral gap γ_G. Let H be a regular undirected graph with d vertices and degree d_0. Then $\mathcal{Z}(G \,|\, H)$ is a regular undirected graph with nd vertices, degree d_0^2 and spectral gap $\gamma_G \gamma_H^2$.*

2.2 Completing the proof

Lemma 5. *Let G be a regular undirected graph with n vertices, degree d^2, and spectral gap γ_G. Let H be a regular undirected graph with d^4 vertices, degree d, and spectral gap H. Then $\mathcal{Z}\big(G^2 \,\big|\, H\big)$ is a regular undirected graph with $d^4 n$ vertices, degree d^2, and spectral gap $\Big(1 - (1-\gamma_G)^2\Big)\gamma_H^2$.*

For (s,t)-connectivity, with some preprocessing, we can assume that G is a regular graph on n vertices with constant degree d^2 and spectral gap $\geq 1/\operatorname{poly}(n)$. We can also assume that there exists a regular expander H on d^4 vertices, degree d, and spectral gap $\geq 3/4$. If we apply Lemma 5 to $\gamma_G \leq 1/16$ and $\gamma_H \geq 3/4$, then $\mathcal{Z}(G^2 \mid H)$ has spectral gap

$$\geq 1.01\gamma_G.$$

That is, we increase the spectral gap by a constant factor! By repeating the construction $O(\log n)$ times, the final graph has constant expansion! The degree, meanwhile, is still d^2 – a constant.

We need to check that the number of vertices did not blow up too much. Each time we apply Lemma 5, the number of vertices increases by d^4. After $O(\log n)$ iterations – if d is a constant – the number of vertices increases by at most $\operatorname{poly}(n)$. Thus we have a constant degree graph with $\operatorname{poly}(n)$ vertices and constant spectral-gap – primed for derandomizing the random walk approach.

All that said, we still need to verify that we can run simulate a random walk on the generated expander in $O(\log n)$ space. Let G_0 denote the input graph (with n vertices and constant degree d^2) and let $G_k = \mathcal{Z}(G_{k-1}^2 \mid H)$ be the graph obtained by the kth iteration of Lemma 5. We claim the following for each index j.

1. *The space required to simulate a step on G_j^2 is $O(1)$ plus the space required to simpulate a step on G_j.*

2. *The space required to simulate a step on $\mathcal{Z}\left(G_j^2 \middle| H\right)$ is $O(1)$ plus the space required to simulate a step on G_j^2.*

If the above hold, then the space required to simulate a step on G_k is $O(k)$, as desired.

Consider the first claim, were we are simulate a step on G_j^2. We are given a vertex v_1 in G_j^2 and two indices $i_1, i_2 \in [d]$. We query (v, i_1) to take a step in G_j, which returns a vertex v_2 in G_j. We then query (v, i_2) to take a step in G_j which returns a vertex w_3. The maximum amount of space we ever use is $O(1)$ plus the space recursively required to take a step in G_j.

Consider the second claim, where we are simulating a step on $\mathcal{Z}\left(G_j^2 \middle| H\right)$. We are given a vertex (v_1, i_1), where v_1 is a vertex in G_j^2 and i_1 is a vertex in H (and at most a constant). We are also given two indices $j_1, j_2 \in [d_0]$. We first take a step in H from i_1 to i_2, using $O(1)$ space. We then query G_j^2 to take i_2th edge from v_1 to v_2 in G_j^2. We then query, for each $i_3 \in [d]$, the i_3th edge from v_2 in G_j^2 until we find that v_1 is the i_3th edge from v_2. Each of these queries take $O(1)$ space plus the space from the recursive call to G_j^2. Finally we use j_2 to update from i_3 to i_4 in H, in $O(1)$ space. The maximum amount of space we ever use is $O(1)$ plus the maximum amount of space recursively used when querying G_j^2.

Up to proving Lemma 4 in the subsequent section, this completes the proof of Theorem 2.

3 Analysis of the zig-zag product

Lemma 4. *Let $G = (V, E)$ be a regular undirected graph with n vertices and degree d, with spectral gap γ_G. Let H be a regular undirected graph with d vertices and degree d_0. Then $\mathcal{Z}(G \mid H)$ is a regular undirected graph with nd vertices, degree d_0^2 and spectral gap $\gamma_G \gamma_H^2$.*

Let R_H be the random walk matrix of H. We can write a step in the zig-zag graph as

$$(I \otimes R_H) Z (I \otimes R_H),$$

where $(I \otimes R_H)$ represents the action where we take a single random step in H but leave the G-coordinate fixed. $(I \otimes R)$ also describes a random step in a certain "tensor product" of graphs that we will more formally discuss in a moment. Z is the (deterministic) zig-zag step that updates the G-coordinate and tranports the H-coordinate, as described above. Ultimately, we want to analyze the spectral gap of $(I \otimes R_H)Z(I \otimes R_H)$.

To give some intuition, recall that in our proof, H will be an expander. That is, taking a few steps in R_H is almost as random as sampling a uniformly random vertex from H. To this end, let $S : \Delta^{V_H} \to \Delta^{V_H}$ be the "random walk" that models sampling a vertex from H uniformly at random in each step. Intuitively, $R_H \approx S$. Suppose we substitute S for R_H in our expression for the random walk in the zig-zag product, giving,

$$(I \otimes S)Z(I \otimes S).$$

This step describes a zig-zag product of G with a different graph (say) H', which is a complete graph with a self-loop at every vertex. Let us walk through a random step in the zig-zag product $\mathcal{Z}(G \,|\, H')$.

1. Starting from $(v_1, i_1) \in G \times [d]$, we first take a random step in H' from i_1 to i_2. By definition of H', $i_2 \in [d]$ is selected uniformly at random.
2. We then move from v_1 to its i_2th neighbor v_2.
3. Then, we move from i_2 to i_3 where v_1 is the i_3th neighbor of v_1.
4. Then we take a step in H' move i_3 to a uniformly index $i_4 \in [d]$.

Overall, we move from (v_1, i_1) to (v_2, i_4) where v_2 is a uniformly random neighbor of v_1, and i_4 is a uniformly random vertex in H'. This is a much simpler step than the zig-zag product on an arbitrary graph, and can be analyzed directly. The second coordinate is essentially just uniformly random noise, and the first coordinate is walking in G. The second coordinate is mathematically irrelevant and the spectral gap is precisely γ_G, the spectral gap of G.

Of course, S is not exactly R_H, and the zig-zag product of G with H' does not decrease the degree as we would like. (In fact, it *increases* the degree.) It remains to quantify the difference between $(I \otimes R_H)Z(I \otimes R_H)$ and $(I \otimes S)Z(I \otimes S)$, which reflects the difference between R_H and S. As we will make more explicit below, the spectral gap of H, γ_H, is also a reflect of the difference between R_H and S. This difference between R_h and S, and the correspondance between the difference and γ_H, is why the spectral gap decreases from γ_G to $\gamma_G \gamma_H^2$.

3.1 Introducing tensor products

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. The **tensor product** of G_1 and G_2, denoted $G_1 \otimes G_2$, is the graph defined as follows. The vertex set is the family $V_1 \times V_2$ of all pairs of vertices from V_1 and V_2,

$$V_1 \times V_2 = \{(v_1, v_2) : v_1 \in V_1,\ v_2 \in V_2\}.$$

For each edge $e_1 = (u_1, v_1)$ and each edge $e_2 = (u_2, v_2)$, we have an edge $((u_1, u_2), (v_1, v_2))$ in $G_1 \otimes G_2$, with multiplicities. That is, an edge $((u_1, u_2), (v_1, v_2))$ in $G_1 \otimes G_2$ has multiplicity equal to the multiplicity of (u_1, v_1) in E_1 times the multiplicity of (u_2, v_2) in E_2.

Theorem 6. *Let G_1 and G_2 be regular undirected graphs with random walk matrices R_1 and R_2 respectively. Then the random walk matrix of $G_1 \otimes G_2$, denoted $R_1 \otimes R_2 : \mathbb{R}^{V_1 \times V_2} \to \mathbb{R}^{V_1 \times V_2}$, is also symmetric. The map*

$$(v_1, v_2) \in \mathbb{R}^{V_1} \times \mathbb{R}^{V_2} \mapsto v_1 \otimes v_2 \in \mathbb{R}^{V_1 \times V_2}$$

gives a one-to-one correspondance between pairs of eigenvectors from G_1 to G_2, where an eigenvector v_1 with eigenvalue λ_1 of G_1 and an eigenvector v_2 with eigenvalue λ_2 of G_2 maps to an eigenvector $v_1 \otimes v_2$ of $G_1 \otimes G_2$ with eigenvalue $\lambda_1 \lambda_2$.

Proof. We have already remarked that the tensor product of two regular vertices is again regular, so the corresponding random walk matrix $R_1 \otimes R_2$ is regular. Let $u_1, \dots, u_{n_1}$ be n orthonormal eigenvectors of R_1 and let $v_1, \dots, v_{n_2}$ be the n orthonormal eigenvectors of R_2. Then by direct inspection one can show that (a) the family $\{u_i \otimes v_j : i \in [n_1], j \in [n_2]\}$ are orthonormal in $\mathbb{R}^{V_1 \times V_2}$ and (b) each $u_i \otimes v_j$ is an eigenvector of $R_1 \otimes R_2$ with eigenvalue $\lambda_i \lambda_j$. ■

3.2 Tensors with the identity

Recall that the following tensor product in particular arises in particular in the zig-zag product $\mathcal{Z}(G \mid V)$:

$$I \otimes R_H.$$

We can think of this as a random step of the following tensor product. Let G_0 be the graph with the same vertex set as G and no edges except for a self-loop at each step. Then a "random step" on G_0 goes no where at all - we stay on the same vertex. In particular, the random walk matrix is I – the identity map on V.

Consider the tensor product $(G_0 \otimes H)$. Each random step on $(G_0 \otimes H)$ consists of a "random" step on G_0 - which just keeps the same coordinate in V - and a random step on H, for which we denote the random walk matrix by R_H. Thus

$$I \otimes R_H,$$

which was was introduced as taking a random step in H and staying put in G - is precisely the random walk matrix.

While tensor products of linear maps are more generally defined, they are easier to describe concretely in the special case where one of the maps is the identity map. Take for example $I \otimes R_H$. Given a vector $x \in \mathbb{R}^{V \times d}$, let $x_u \in \mathbb{R}^d$ denote the "u-slice" of x defined by

$$(x_u)(i) = x(u, i).$$

We can think of $I \otimes R_H$ as applying R_H to each slice; i.e.,

$$((I \otimes R_H)x)_u = R_H x_u.$$

More generally, if $I : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ is the identity map, $A : \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ is any linear map, and $i \in [n_1]$, then we have

$$((I \otimes A)x)_i = A x_i.$$

Form this definition we get the following useful algebraic rule:

$$(I \otimes A + B) = (I \otimes A) + (I \otimes B),$$

since

$$((I \otimes A + B)x)_i = (A + B)x_i = Ax_i + Bx_i = ((I \otimes A)x)_i + ((I \otimes B)x)_i.$$

3.3 A helpful inequality

Lemma 7. *The* **operator norm**

$$\|A\| = \sup\{\|Ax\| : \|x\| = 1\}.$$

Lemma 8. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a symmetric and positive semi-definite linear map, and let $\lambda \geq 0$ be the maximum absolute value of any eigenvalue of A. Then*

$$\|A\| = \lambda.$$

Proof. Recall that

$$\lambda = \max_x \left| \frac{\langle x, Ax \rangle}{\langle x, x \rangle} \right|,$$

and by the spectral theorem,

$$\lambda^2 = \max_x \frac{\langle x, A^2 x \rangle}{\langle x, x \rangle}.$$

On the other hand,

$$\|A\|^2 = \sup_{\|x\|=1} \langle Ax, Ax \rangle = \sup_{\|x\|=1} \langle x, A^2 x \rangle = \lambda^2.$$

■

Lemma 9. *Let A be a random walk matrix with stationary distribution u. Then for any non-dominant eigenvalue $\lambda \neq 1$, we have*

$$\lambda \leq \|A\|.$$

3.4 Zig-Zags and tensor products

Let R_H be the random walk matrix of H. Recall previously that we applied the spectral theorem to R_H, indirectedly by similarity to the normalized random walk matrix, to derive very precise bounds on the converge rate towards a random walk, as a function of the spectral gap. Here to we want to analyze the eigenvalues of the zig-zag product, and we start from reanalyzing R_H. In some ways our analysis is even easier this time because when G is a regular graph with constant degree d, $R_H = \frac{1}{d}A$ is already a symmetric map! (Here d is an integer, and not a vector). By the spectral theorem for symmetric maps, combined the the Perron-Frobenius theorem for random walks, we have

$$R_H = u_1 \otimes u_1 + \lambda_2(u_2 \otimes u_2) + \cdots + \lambda_n(u_n \otimes u_n),$$

where $u_1, \ldots, u_n \in \mathbb{R}^d$ forms an orthonormal bases and $\lambda_2, \ldots, \lambda_n \in [1 - \gamma_H, \gamma_H - 1]$. Recall that because R_H is regular, the stationary [distribution is the uniform distribution, $\mathbb{1}/d$. Since $R_H \mathbb{1} = \mathbb{1}$, the first eigenvector u_1 must be (proportional to) $\mathbb{1}$. This gives

$$R_H = \frac{1}{d}(\mathbb{1} \otimes \mathbb{1}) + \lambda_2(u_2 \otimes u_2) + \cdots + \lambda_n(u_n \otimes u_n).$$

Consider the first term $\frac{1}{d}(\mathbb{1} \otimes \mathbb{1})$. This map sends any distribution to the stationary one, $\mathbb{1}/d$ – that is, this is exactly the map S! Thus we have

$$R_H = \frac{1}{d}(\mathbb{1} \otimes \mathbb{1}) + \lambda_2(u_2 \otimes u_2) + \cdots + \lambda_n(u_n \otimes u_n),$$

where $\mathbb{1}/\sqrt{d}$, $u_2, \ldots, u_n$ form an orthonormal basis and $\lambda_2, \ldots, \lambda_n \in [1 - \gamma_H, \gamma_H - 1]$. Let us decompose R_H as

$$R_H = S + T \text{ where } S = \frac{1}{d}(\mathbb{1} \otimes \mathbb{1}) \text{ and } T = \lambda_2(u_2 \otimes u_2) + \cdots + \lambda_n(u_n \otimes u_n).$$

Let

$$R'_H = R_H - \gamma_H S;$$

then R'_H has all its eigenvalues in the range $[1 - \gamma_H, \gamma_H - 1]$.

Let us now return to the zig-zag product and substitute in for R_H. We have

$$\begin{aligned}
&(I \otimes R_H)Z(I \otimes R_H) = \big(I \otimes \gamma_H S + R'_H\big)Z\big(I \otimes \gamma_H S + R'_H\big) \\
&= \big(\gamma_H(I \otimes S) + \big(I \otimes R'_H\big)\big)Z\big(\gamma_H(I \otimes S) + \big(I \otimes R'_H\big)\big) \\
&= \gamma_H^2(I \otimes S)Z(I \otimes S)Z + \gamma_H\big((I \otimes S)Z\big(I \otimes R'_H\big) + \big(I \otimes R'_H\big)Z(I \otimes S)\big) + \big(I \otimes R'_H\big)Z\big(I \otimes R'_H\big).
\end{aligned}$$

Let us first analyze the last 3 terms.

Let $x \in \mathbb{R}^{V \times d}$ be any unit vector orthogonal to the uniform distribution (which is the first eigenvector). We have

$$\big\langle z, (I \otimes S)Z\big(I \otimes R'_H\big)z\big\rangle = \big\|(I \otimes S)Z\big(I \otimes R'_H\big)x\big\| \le \|(I \otimes S)\|\|Z\|\big\|\big(I \otimes R'_H\big)\big\| \le 1 - \gamma_H.$$

Likewise the second term contributes $1 - \gamma_H$ (times another γ_H) and the third term contributes $(1 - \gamma_H)^2$.

For the finale of our analysis, consider the remaining term, $(I \otimes S)Z(I \otimes S)$. As observed ealier, we have

$$(I \otimes S)Z(I \otimes S) = (A \otimes S) \qquad (!)$$

Note that $(A \otimes S)$ is the tensor product of G and H with has the same stationary distribution; namely the uniform distribution. Moreover, $(A \otimes S)$ has the same eigenvalues as A with the same multiplicity, since S has only one eigenvector with eigenvalue 1 and the rest are all 0. In particular, for any vector $x \in \mathbb{R}^{V \times d}$ orthogonal to the stationary distribution on $V \times [d]$, we have

$$|\langle x, (I \otimes S)Z(I \otimes S)x\rangle| = |\langle x, (A \otimes S)x\rangle| \le 1 - \gamma_G.$$

Putting everything together gives the desired bound. Viola!

References

[RVW02] Omer Reingold, Salil Vadhan, and Avi Wigderson. “Entropy Waves, the Zig-Zag Graph Product, and New Constant-Degree Expanders”. In: *Annals of Mathematics*. Second Series 155.1 (2002), pp. 157–187. URL: `http://www.jstor.org/stable/3062153`.

[Rei08] Omer Reingold. “Undirected connectivity in log-space”. In: *J. ACM* 55.4 (2008), 17:1–17:24. Preliminary version in STOC, 2005.

[Vad12] Salil P. Vadhan. “Pseudorandomness”. In: *Found. Trends Theor. Comput. Sci.* 7.1-3 (2012), pp. 1–336.